

# Maximal quantum mechanical symmetry: Projective representations of the Weyl-Heisenberg automorphism group

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## Abstract

The Heisenberg commutation relations are fundamental to quantum mechanics. Consider a symmetry group and its projective representations transforming quantum physical states. The Heisenberg commutation relations must be valid in the original and in the transformed states. This paper determines the maximal symmetry group and the projective representations acting on the quantum physical states that so that the Heisenberg commutation relations are valid in all the transformed states. Any symmetry that is physical in the sense that it preserves the Heisenberg commutation relations must be a subgroup and its projective representations must be contained in the above maximal symmetry representations.

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## 1. Introduction

The Heisenberg commutation relations are fundamental to quantum mechanics,

$$[\hat{P}_i, \hat{Q}_j] = i\hbar\delta_{i,j}\mathbf{1}, \quad (1)$$

where  $i, j, \dots = 1, \dots, n$ . The hermitian operators  $\hat{P}_i$  and  $\hat{Q}_j$  represent quantum mechanical momentum and position observables acting on states  $|\psi\rangle$  that are elements of a Hilbert space  $\mathbf{H}$  for which  $\mathbf{1}$  is the unit operator. (We will use natural units for which  $\hbar = 1$  throughout the paper.)

A basic assumption of quantum mechanics is that the Heisenberg commutation relations are valid when acting on any physical state. Physical states in quantum mechanics are rays  $\Psi$  that are equivalence classes of states  $|\psi\rangle$  in the Hilbert space that are equal up to a phase [1],

$$\Psi = [|\psi\rangle], \quad |\tilde{\psi}\rangle \simeq |\psi\rangle \Leftrightarrow |\tilde{\psi}\rangle = e^{i\theta} |\psi\rangle. \quad (2)$$

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The reason for this is that physically observable probabilities are given by the square of the modulus of the states. The square of the modulus is the same for any representative state in the ray,

$$P(\alpha \rightarrow \beta) = |(\Psi_\beta, \Psi_\alpha)|^2 = |\langle \tilde{\psi}_\beta | \tilde{\psi}_\alpha \rangle|^2 = |\langle \psi_\beta | \psi_\alpha \rangle|^2. \quad (3)$$

Symmetry transformations between physical states (i.e. rays  $\Psi$ ) are given by operators  $U$  that leave invariant the square of modulus,

$$|(U\Psi_\beta, U\Psi_\alpha)|^2 = |(\Psi_\beta, \Psi_\alpha)|^2. \quad (4)$$

These transformations  $U$  are the representation of a group

$$\varrho : \mathcal{G} \rightarrow U(\mathbf{H}) : g \mapsto U = \varrho(g). \quad (5)$$

Theorem 2 in Appendix A states that any representation of any connected group<sup>1</sup> that leaves invariant the square of the modulus is always equivalent to a unitary and linear operator mapping the Hilbert space  $\mathbf{H}$  into itself,  $U \in U(\mathbf{H})^2$ . This unitary operator also acts on any representative in the equivalence class of states that defines the ray,

$$\Psi' = U\Psi, \quad |\psi'\rangle = U|\psi\rangle, \quad U^\dagger = U^{-1}, \quad U \in U(\mathbf{H}). \quad (6)$$

The representations  $\varrho$  are referred to as projective representations. If  $\mathcal{G}$  is a connected Lie group, the fundamental Theorem 2 states that these projective representations are equivalent to the ordinary unitary representations  $\nu$  of the central extension  $\tilde{\mathcal{G}}$  of  $\mathcal{G}$ .

Under these unitary transformations of states, the hermitian operators in (1) transform as

$$U\hat{P}_i|\psi\rangle = U\hat{P}_iU^{-1}U|\psi\rangle = \hat{P}'_i|\psi'\rangle, \quad (7)$$

where

$$\hat{P}'_i = U\hat{P}_iU^{-1}, \quad \hat{Q}'_i = U\hat{Q}_iU^{-1}, \quad \mathbf{1}' = U\mathbf{1}U^{-1} = \mathbf{1}. \quad (8)$$

The requirement that the Heisenberg commutation relations are valid when acting on any physical state then requires that the transformed operators also satisfy the Heisenberg commutation relations,

$$[\hat{P}'_i, \hat{Q}'_i] = i\hbar\delta_{i,j}\mathbf{1}. \quad (9)$$

These Heisenberg commutation relations are not satisfied for all unitary transformations  $U$  acting on a physical state. While every transformation between physical states is given by a unitary representation of a group, not every unitary representation of a group defines a transformation between physical states. If the unitary transformation results in states in which the fundamental quantum mechanical Heisenberg commutation relations do not hold, the state is not physical. Therefore, the requirement that the Heisenberg commutation relations are satisfied for the transformed states *is* a condition on the admissible group of transformations.

The problem that this paper addresses is to determine this maximal group and its projective representations that are transformations between physical states such that the Heisenberg commutation relations hold.

<sup>1</sup>A ‘connected group’ is a group for which all of its elements are path connected to the identity element.

<sup>2</sup>The theorem also states that if the group is not connected, the representations may be anti-unitary and anti-linear.

The key to solving this is to note that the Heisenberg commutation relations themselves are the hermitian representation of the Lie algebra of the Weyl-Heisenberg Lie group  $\mathcal{H}(n)$  that correspond to the unitary representations of this group [2].

The Lie algebra of the Weyl-Heisenberg Lie group is the semidirect product<sup>3</sup> of two abelian groups [3]

$$\mathcal{H}(n) \simeq \mathcal{A}(n) \otimes_s \mathcal{A}(n+1), \quad (10)$$

where  $\mathcal{A}(m)$  is the abelian Lie group isomorphic to the reals under addition,  $\mathcal{A}(m) \simeq (\mathbb{R}^m, +)$ . Its Lie algebra is given by

$$[P_i, Q_i] = \delta_{i,j} I, \quad [P_i, I] = 0, [Q_i, I] = 0. \quad (11)$$

The operators are given by the hermitian representation  $v'$  of this algebra that is the lift of the unitary representation  $v$  of the group,  $v' = T_e v$ ,

$$\hat{P}_i = v'(P_i), \hat{Q}_i = v'(Q_i), \hat{I} = v'(I). \quad (12)$$

The hermitian representations of the Lie algebra corresponding to the faithful unitary irreducible representations of  $\mathcal{H}(n)$  are given

$$v'([P_i, Q_i]) = [v'(P_i), v'(Q_i)] = i \delta_{i,j} v'(I) = i \lambda \delta_{i,j} \mathbf{1}. \quad (13)$$

The hermitian representation of the central generator  $I$  is  $v'(I) = \lambda \mathbf{1}$  due to Schur's lemma. The  $\lambda \in \mathbb{R}$  label the irreducible representations.  $\lambda \neq 0$  are faithful representations. For a simply connected group,  $g = e^X$  and  $v(g) = e^{iv'(X)}$ . Then if  $v(g)$  is unitary,  $v'(X)$  is hermitian. The insertion of the  $i$  in the exponent to make  $v'(X)$  hermitian rather than anti-hermitian requires it to also be inserted in the commutation relations. The irreducible representation of the algebra (13) with  $\lambda = 1$  are the Heisenberg commutation relations (1). (We are using natural units in which  $\hbar = 1$ ).

The transformed hermitian generators may now be written as

$$\begin{aligned} \hat{P}'_i &= v(g) v'(P_i) v(g)^{-1} = v'(P'_i), \\ \hat{Q}'_i &= v(g) v'(Q_i) v(g)^{-1} = v'(Q'_i), \end{aligned} \quad (14)$$

where  $g \in \check{\mathcal{G}}$  and

$$P'_i = g P_i g^{-1}, Q'_i = g Q_i g^{-1}, I' = g I g^{-1} = I. \quad (15)$$

The transformed Heisenberg commutation relations are

$$[\hat{P}'_i, \hat{Q}'_i] = v'(g[P_i, Q_i]g^{-1}) = i \delta_{i,j} v'(g I g^{-1}) = i \delta_{i,j} \mathbf{1}. \quad (16)$$

As the representation is faithful (i.e. an isomorphism), this is equivalent to the Lie algebra condition

$$[P'_i, Q'_i] = g[P_i, Q_i]g^{-1} = \delta_{i,j} g I g^{-1} = \delta_{i,j} I. \quad (17)$$

The group that leaves the Lie algebra of the Weyl-Heisenberg group invariant is the automorphism group of the algebra. As  $\mathcal{H}(n)$  is simply connected, this is the same as the automorphism

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<sup>3</sup>In our notation for a semidirect product  $\mathcal{G} \simeq \mathcal{K} \otimes_s \mathcal{N}$ ,  $\mathcal{N}$  is the normal subgroup (see Appendix A). Also,  $\mathcal{A} \simeq \mathcal{B}$  is the notation for a group isomorphism.

group,  $\mathcal{A}ut_{\mathcal{H}(n)}$ , of the Weyl-Heisenberg group  $\mathcal{H}(n)$ . This is precisely the maximal group leaving the Weyl-Heisenberg equations invariant that we seek. Any quantum symmetry must be a subgroup of this group and contained in these representations.

The problem that this paper addresses is to determine this group and these representations. We will show that the automorphism group of the Weyl-Heisenberg group is [3]

$$\mathcal{A}ut_{\mathcal{H}(n)} \simeq \mathcal{D} \otimes_s \mathcal{H}\overline{Sp}(2n), \quad (18)$$

where

$$\mathcal{H}\overline{Sp}(2n) \simeq \overline{Sp}(2n) \otimes_s \mathcal{H}(n), \quad \mathcal{D} \simeq \mathbb{Z}_2 \otimes \mathcal{D}^+, \quad (19)$$

with  $\mathbb{Z}_2 \simeq \{1, -1\}$  and  $\mathcal{D} \simeq (\mathbb{R}^+, \times)$ . Furthermore it is given by the central extension,  $\mathcal{H}\overline{Sp}(2n) \simeq \widetilde{ISp}(2n)$  that is defined by the short exact sequence

$$e \rightarrow \mathbb{Z} \otimes \mathcal{A}(1) \rightarrow \mathcal{H}\overline{Sp}(2n) \rightarrow ISp(2n) \rightarrow e. \quad (20)$$

$\mathbb{Z}$  is the center of  $\overline{Sp}(2n)$  and  $\mathcal{A}(1)$  is the center of  $\mathcal{H}(n)$ .  $ISp(2n)$  is the inhomogeneous symplectic group familiar from classical Hamiltonian mechanics,

$$ISp(2n) \equiv Sp(2n) \otimes_s \mathcal{A}(2n). \quad (21)$$

The group  $\mathcal{D}$  is not connected and has two components,  $\mathcal{D}/\mathcal{D}^+ \simeq \mathbb{Z}_2$  where  $\mathcal{D}^+$  is the component connected to the identity. It follows that the automorphism group has two components that are not connected with

$$\mathcal{A}ut_{\mathcal{H}(n)}/\mathcal{A}ut_{\mathcal{H}(n)}^c \simeq \mathbb{Z}_2 \quad (22)$$

where the component connected to the identity is

$$\mathcal{A}ut_{\mathcal{H}(n)}^c \simeq \mathcal{D}^+ \otimes_s \mathcal{H}\overline{Sp}(2n). \quad (23)$$

The automorphism group has the required property that it is maximally centrally extended,

$$\mathcal{A}ut_{\mathcal{H}(n)}^c \simeq \widetilde{\mathcal{A}ut}_{\mathcal{H}(n)}^c, \quad (24)$$

and we will show that this also extends to the full automorphism group  $\mathcal{A}ut_{\mathcal{H}(n)}$ .

## 2. The Weyl-Heisenberg group

The Weyl-Heisenberg Lie group is defined to be the semi-direct product of two abelian groups as given in (10). Therefore, it has an underlying manifold diffeomorphic to  $\mathbb{R}^{2n+1}$  and is simply connected. In a global coordinate system  $p, q \in \mathbb{R}^n, \iota \in \mathbb{R}$ , the group product and inverse of the Weyl-Heisenberg group may be written

$$\Upsilon(p', q', \iota') \Upsilon(p, q, \iota) = \Upsilon(p' + p, q' + q, \iota + \iota' + \frac{1}{2}(p' \cdot q - q' \cdot p)), \quad (25)$$

$$\Upsilon(p, q, \iota)^{-1} = \Upsilon(-p, -q, -\iota). \quad (26)$$

The identity element is  $e = \Upsilon(0, 0, 0)$ .

We need to verify that these group product and inverse relations result in the semidirect product given by (10). First, these expressions enable us to identify the abelian subgroups

$$\Upsilon(0, q, \iota) \in \mathcal{A}(n+1), \Upsilon(p, 0, 0) \in \mathcal{A}(n). \quad (27)$$

as, using (25-26), these satisfy the group product relations

$$\Upsilon(0, q'', \iota'') = \Upsilon(0, q', \iota') \Upsilon(0, q, \iota) = \Upsilon(0, q' + q, \iota + \iota'), \quad (28)$$

$$\Upsilon(p'', 0, 0) = \Upsilon(p', 0, 0) \Upsilon(p, 0, 0) = \Upsilon(p' + p, 0, 0). \quad (29)$$

Additional abelian subgroups are likewise given by

$$\Upsilon(p, 0, \iota) \in \mathcal{A}(n+1), \quad \Upsilon(0, q, 0) \in \mathcal{A}(n+1) \quad (30)$$

We calculate the inner automorphisms of the group using (25-26) to be<sup>4</sup>

$$\begin{aligned} \varsigma_{\Upsilon(p', q', \iota')} \Upsilon(p, q, \iota) &= \Upsilon(p', q', \iota') \Upsilon(p, q, \iota) \Upsilon(p', q', \iota')^{-1} \\ &= \Upsilon(p, q, \iota + p'q - q' \cdot p). \end{aligned} \quad (31)$$

In particular, note that for each of the choices of the subgroups

$$\varsigma_{\Upsilon(p', q', \iota')} \Upsilon(0, q, \iota) = \Upsilon(0, q, \iota + p'q), \quad (32)$$

$$\varsigma_{\Upsilon(p', q', \iota')} \Upsilon(p, 0, \iota) = \Upsilon(p, 0, \iota - q' \cdot p). \quad (33)$$

This means that both of the  $\mathcal{A}(n+1)$  subgroups given in (27), (30) are normal subgroups. Another special case of (25) is

$$\varsigma_{\Upsilon(0, 0, \iota')} \Upsilon(p, q, \iota) = \Upsilon(p, q, \iota). \quad (34)$$

and therefore the elements  $\Upsilon(0, 0, \iota')$  commute with all elements of the group. Furthermore, these are the only elements that commute with all other elements of the group. Therefore the  $\mathcal{A}(1)$  group that is defined by the elements  $\Upsilon(0, 0, \iota)$  is the center of the group,  $\mathcal{Z} \simeq \mathcal{A}(1)$ .

The final step to verify that the group relations defined by (25-26) results in the Weyl-Heisenberg group having the structure of a semidirect product given in (10). We have already established that there are two choices for the  $\mathcal{A}(n)$  subgroup and  $\mathcal{A}(n+1)$  normal subgroup. It is clear in both cases that

$$\mathcal{A}(n) \cap \mathcal{A}(n+1) = \mathbf{e}, \quad (35)$$

as the identity  $\Upsilon(0, 0, 0)$  is the only element in both groups for both cases. It remains to show that  $\mathcal{A}(n+1)\mathcal{A}(n) \simeq \mathcal{H}(n)$ . Using the group product (25), for each of the cases (27), (30), this is

$$\Upsilon(0, q, \iota) \Upsilon(p, 0, 0) = \Upsilon(p, q, \iota - \frac{1}{2}q \cdot p), \quad (36)$$

$$\Upsilon(p, 0, \iota) \Upsilon(0, q, 0) = \Upsilon(p, q, \iota + \frac{1}{2}p \cdot q). \quad (37)$$

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<sup>4</sup>We always use  $\varsigma$  to define the similarity map  $\varsigma_g h \equiv ghg^{-1}$  in what follows.

The map

$$\varphi^\pm : \mathcal{H}(n) \rightarrow \mathcal{H}(n) : \Upsilon(p, q, \iota) \mapsto \Upsilon^\pm(p, q, \iota^\pm) = \Upsilon(p, q, \iota \mp \frac{1}{2}p \cdot q) \quad (38)$$

is a homomorphism that is onto and the kernel is trivial. Therefore, the map  $\varphi^\pm$  is an isomorphism and the Weyl-Heisenberg group has the semidirect product structure given in (10) for either of the choices of abelian subgroup given by (27), (30).

The isomorphisms  $\Upsilon^\pm(p, q, \iota^\pm)$  are referred to as the polarized realizations. Their group products are computed directly from (25-26) to be

$$\begin{aligned} \Upsilon^+(p', q', \iota') \Upsilon^+(p, q, \iota) &= \Upsilon^+(p' + p, q' + q, \iota + \iota + p' \cdot q), \\ \Upsilon^-(p', q', \iota') \Upsilon^-(p, q, \iota) &= \Upsilon^-(p' + p, q' + q, \iota + \iota - q' \cdot p), \\ \Upsilon^\pm(p, q, \iota)^{-1} &= \Upsilon^\pm(-p, -q, -\iota \pm p \cdot q). \end{aligned} \quad (39)$$

Note that the polarized realizations factor directly

$$\Upsilon^+(0, q, \iota) \Upsilon^+(p, 0, 0) = \Upsilon^+(p, q, \iota), \quad (40)$$

$$\Upsilon^-(p, 0, \iota) \Upsilon^-(0, q, 0) = \Upsilon^-(p, q, \iota). \quad (41)$$

The existence of the isomorphisms  $\varphi^\pm$  and these two different normal subgroups  $\mathcal{A}(n+1)$  with elements  $\Upsilon(p, 0, \iota)$  and  $\Upsilon(0, q, \iota)$  whose intersection is the center  $\mathcal{Z} \simeq \mathcal{A}(1)$  is responsible for many of the remarkable properties of the Weyl-Heisenberg group. In fact, we shall see shortly that the choice of the normal subgroup in determining the unitary representations when applying the Mackey theorems results in unitary representations with either  $p$  or  $q$  diagonal.

The Lie group is a matrix group and may be realized by the  $2n+2$  dimensional square matrices

$$\Upsilon(p, q, \iota) = \begin{pmatrix} 1_n & 0 & 0 & p \\ 0 & 1_n & 1 & q \\ q^t & -p^t & 1 & 2\iota \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (42)$$

$1_m$  denotes the unit matrix in  $m$  dimensions and the  $t$  superscript denotes the transpose. The group multiplication and inverse (25-26) are realized by matrix multiplication and inverse. The matrix realization corresponds to a coordinate system of the Lie group and is therefore not unique. The polarized matrix realizations are given by the  $n+2$  dimensional square matrices

$$\Upsilon^+(p, q, \iota) = \begin{pmatrix} 1 & q^t & \iota \\ 0 & 1_n & p \\ 0 & 0 & 1 \end{pmatrix}, \quad \Upsilon^-(p, q, \iota) = \begin{pmatrix} 1 & p^t & \iota \\ 0 & 1_n & q \\ 0 & 0 & 1 \end{pmatrix}. \quad (43)$$

The Lie algebra of the Weyl-Heisenberg group may be computed from any of these realizations. In all cases the coordinates are nonsingular at the origin and therefore, choosing the unpolarized form, the generators are given by

$$Q_i = \frac{\partial}{\partial p^i} \Upsilon(p, q, \iota)|_e, P_i = \frac{\partial}{\partial q^i} \Upsilon(p, q, \iota)|_e, I = \frac{\partial}{\partial \iota} \Upsilon(p, q, \iota)|_e. \quad (44)$$

A general element of the algebra is then

$$Z = p^i Q_i + q^i P_i + \iota I. \quad (45)$$

The nonzero commutation relations are

$$[P_i, Q_i] = \delta_{i,j} I. \quad (46)$$

$I$  is a central generator.

It is convenient to also introduce the notation that combines the  $p, q \in \mathbb{R}^n$  into a single  $2n$  tuple  $z = (p, q)$ ,  $z \in \mathbb{R}^{2n}$ . Then the group product and inverse are

$$\Upsilon(z', \iota') \Upsilon(z, \iota) = \Upsilon(z' + z, \iota + \iota - \frac{1}{2} z'^t \zeta z), \quad \Upsilon(z, \iota)^{-1} = \Upsilon(-z, -\iota) \quad (47)$$

and the unpolarized matrix realization is

$$\Upsilon(z, \iota) = \begin{pmatrix} 1_{2n} & 0 & z \\ -z^t \zeta & 1 & 2\iota \\ 0 & 0 & 1 \end{pmatrix}, \quad \zeta = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}. \quad (48)$$

The Lie algebra has general element  $Z = z^\alpha Z_\alpha + \iota I$ ,  $\alpha, \beta, \dots = 1, \dots, 2n$  that satisfy the nonzero commutation relations

$$[Z_\alpha, Z_\beta] = \zeta_{\alpha\beta} I. \quad (49)$$

### 3. The automorphism group of the Weyl-Heisenberg group

The automorphism group  $\mathcal{Aut}_{\mathcal{G}}$  of a Lie group  $\mathcal{G}$  is the group of all isomorphisms  $\varsigma_{\Omega}$  of the group onto itself.

$$\varsigma_{\Omega} : \mathcal{G} \rightarrow \mathcal{G} : g \mapsto g' = \varsigma_{\Omega} g = \Omega g \Omega^{-1} \quad (50)$$

where  $g \in \mathcal{G}$  and  $\Omega \in \mathcal{Aut}_{\mathcal{G}}$ . The inner automorphisms are always automorphisms and therefore  $\mathcal{G}$  is a normal subgroup of the automorphism group,  $\mathcal{G} \subset \mathcal{Aut}_{\mathcal{G}}$ . The automorphisms of the group induce automorphisms on the Lie algebra  $\mathfrak{g}$  of  $\mathcal{G}$ ,

$$\varsigma'_{\Omega} : \mathfrak{a}(\mathcal{G}) \rightarrow \mathfrak{a}(\mathcal{G}) : X \mapsto \tilde{X} = \varsigma_{\Omega} X = \Omega X \Omega^{-1} = X + \frac{1}{2} [Y, X] + \dots \quad (51)$$

where  $\Omega = e^Y$  on some neighborhood of the identity where  $Y$  is an element of the Lie algebra  $\mathfrak{a}(\mathcal{G})$  of  $\mathcal{G}$ . If the group  $\mathcal{G}$  is simply connected, then the automorphism groups of the group and algebra are isomorphic.

This may be applied to the Weyl-Heisenberg algebra. A general linear transformation on a basis of the Weyl-Heisenberg algebra is

$$\tilde{Z}_\alpha = d_\alpha^\beta Z_\beta + z^\alpha I, \quad \tilde{I} = c^\alpha Z_\alpha + dI. \quad (52)$$

For this to be an automorphism, they must satisfy the commutation relations,

$$0 = [\tilde{I}, \tilde{I}] = c^\alpha c^\beta [Z_\alpha, Z_\beta], \quad (53)$$

$$0 = [\tilde{Z}_\alpha, \tilde{I}] = d_\alpha^\beta c^\delta [Z_\beta, Z_\delta] = \frac{1}{d} a_\alpha^\delta c^\gamma \zeta_{\delta\gamma} (\tilde{I} - c^\epsilon Z_\epsilon), \quad (54)$$

$$\zeta_{\alpha\beta} \tilde{I} = [\tilde{Z}_\alpha, \tilde{Z}_\beta] = a_\alpha^\delta a_\beta^\gamma [Z_\delta, Z_\gamma] = \frac{1}{d} a_\alpha^\delta a_\beta^\gamma \zeta_{\delta\gamma} (\tilde{I} - c^\epsilon Z_\epsilon). \quad (55)$$

This has the solution

$$c^\gamma = 0, \quad a_\alpha^\delta = \epsilon \delta \Sigma_\alpha^\delta, \quad d = \delta^2, \quad (56)$$

where  $z \in \mathbb{R}^{2n}$ ,  $\delta \in \mathbb{R}^+$ ,  $\epsilon = \pm 1 \in \mathbb{Z}_2$  and  $\Sigma^\dagger \zeta \Sigma = \zeta$  and so  $\Sigma \in Sp(2n)$ .

While this gives the general form of the automorphism, it does not give us the actual structure of the automorphism group. This may also be determined using

$$\varsigma_\Omega \Upsilon(z', \iota') = \Upsilon(z'', \iota'') \Rightarrow \Omega \Upsilon(z', \iota') = \Upsilon(z'', \iota'') \Omega \quad (57)$$

where  $\Omega$  is a  $2n + 2$  dimensional square matrix and  $\Upsilon(z, \iota)$  is the matrix realization of the Weyl-Heisenberg group given in (48). We can write  $\Omega$  in terms of the submatrices

$$\Omega = \begin{pmatrix} a & c & z \\ f & d & j \\ g & h & \epsilon \end{pmatrix} \quad (58)$$

where  $j, d, r, h, e \in \mathbb{R}$ ,  $c, w, f, g \in \mathbb{R}^{2n}$  and  $a$  is a  $2n$  dimensional square submatrices and then solve (57) to obtain

$$\Omega(\epsilon, \delta, \Sigma, z, \iota) = \begin{pmatrix} \epsilon \delta \Sigma & 0 & z \\ -\epsilon \delta z^\dagger \zeta \Sigma & \delta^2 & 2\iota \\ 0 & 0 & 1 \end{pmatrix} \quad (59)$$

where  $z \in \mathbb{R}^{2n}$ ,  $\delta \in \mathbb{R}^+$ ,  $\epsilon = \pm 1 \in \mathbb{Z}_2$ ,  $\iota \in \mathbb{R}$  and  $\Sigma^\dagger \zeta \Sigma = \zeta$  and so  $\Sigma \in Sp(2n)$ .

Comparing this matrix with (48), it is clear that the elements of the subgroup  $\mathcal{H}(n)$  of inner automorphisms are

$$\Upsilon(z, \iota) = \Omega(1, 1, 1_{2n}, z, \iota), \quad \Upsilon(z, \iota) \in \mathcal{H}(n). \quad (60)$$

Direct matrix multiplication shows that the  $\Omega$  define a group with product and inverse

$$\begin{aligned} \Omega(\epsilon'', \delta'', \Sigma'', z'', \iota'') &= \Omega(\epsilon', \delta', \Sigma', z', \iota') \Omega(\epsilon, \delta, \Sigma, z, \iota) \\ &= \Omega(\epsilon' \epsilon, \delta' \delta, \Sigma' \Sigma, z' + \epsilon' \delta' \Sigma' z, \iota' + \delta'^2 \iota - \frac{1}{2} \epsilon' \delta' z'^\dagger \zeta \Sigma' z), \end{aligned} \quad (61)$$

$$\Omega(\epsilon, \delta, \Sigma, z, \iota)^{-1} = \Omega(\epsilon, \delta^{-1}, \Sigma^{-1}, -\epsilon \delta^{-1} \Sigma^{-1} z, -\delta^{-2} \iota). \quad (62)$$

Using these relations, we can straightforwardly show that the  $\Omega(\epsilon, \delta, \Sigma, w, r)$  are elements of a group with two components that are not connected that are a semidirect product

$$\text{aut}_{\mathcal{H}(n)} \simeq \mathbb{Z}_2 \otimes_s \text{aut}_{\mathcal{H}(n)}^c. \quad (63)$$

$\text{aut}_{\mathcal{H}(n)}^c$  is the component connected to the identity and  $\text{aut}_{\mathcal{H}(n)}$  is the extended group with two components that are not connected. The group  $\text{aut}_{\mathcal{H}(n)}$  is a semidirect product given by

$$\text{aut}_{\mathcal{H}(n)} \simeq (\mathcal{D} \otimes Sp(2n)) \otimes_s \mathcal{H}(n) \simeq \mathcal{D} \otimes_s Sp(2n) \otimes_s \mathcal{H}(n). \quad (64)$$

The elements of the subgroups in these semidirect product decompositions are

$$\Omega(\epsilon, 1, 1_{2n}, 0, 0) \in \mathbb{Z}_2, \Omega(1, \delta, 1_{2n}, 0, 0) \in \mathcal{D} \simeq (\mathbb{R}^+, \times), \quad (65)$$

$$\Omega(1, 1, 1_{2n}, z, \iota) = \Upsilon(z, \iota) \in \mathcal{H}(n), \Omega(1, 1, \Sigma, 0, 0) \simeq \Sigma \in Sp(2n). \quad (66)$$

The semidirect product in (63) is established by noting that

$$\mathbb{Z}_2 \cap \text{aut}_{\mathcal{H}(n)} = \{\Omega(1, 1, 1_{2n}, 0, 0)\} = e, \quad (67)$$

$$\Omega(1, \delta, \Sigma, \omega, \iota) \Omega(\epsilon, 1, 1_{2n}, 0, 0) = \Omega(\epsilon, \delta, \Sigma, \omega, \iota). \quad (68)$$



The semidirect product in (64) is established using (61) by noting that

$$\mathcal{D} \cap Sp(2n) \simeq \mathbf{e}, (\mathcal{D} \otimes Sp(2n)) \cap \mathcal{H}(n) \simeq \mathbf{e}, \quad (69)$$

$$\Omega(1, \delta, 1_{2n}, 0, 0) \Omega(1, 1, \Sigma, 0, 0) = \Omega(1, 1, \Sigma, 0, 0) \Omega(1, \delta, 1_{2n}, 0, 0), \quad (70)$$

$$\Upsilon(z, \iota) \Omega(1, \delta, \Sigma, 0, 0) = \Omega(1, \delta, \Sigma, z, \iota). \quad (71)$$

Direct computation using (61-62) results in the automorphisms of the Weyl-Heisenberg normal subgroup given by

$$\varsigma_{\Omega(\epsilon', \delta', \Sigma', z', \iota')} \Upsilon(z, \iota) = \Upsilon(\epsilon' \delta' \Sigma' z, \delta'^2 \iota + \frac{1}{2} \epsilon' \delta' ((\Sigma' z)^\dagger \zeta z' - z'^\dagger \zeta \Sigma' z)). \quad (72)$$

This establishes that  $\mathcal{H}(n)$  is a normal subgroup of  $\text{aut}_{\mathcal{H}(n)}$ . In the same manner, it may be shown that  $\text{aut}_{\mathcal{H}(n)}^c$  is a normal subgroup of  $\text{aut}_{\mathcal{H}(n)}$ . This completes the verification of the semidirect product structures in (63) and (64).

The central extension of a connected automorphism group of a group  $\mathcal{G}$  is centrally extended as the central elements commute with all elements of the group and therefore are also automorphisms of  $\mathcal{G}$ . The maximal connected automorphism group is therefore fully centrally extended. If the automorphism group is not connected the central extension is not necessarily unique. Therefore these must be addressed on a case by case basis.

We therefore consider first the connected component  $\mathcal{Aut}_{\mathcal{H}(n)}^c$  of the Weyl-Heisenberg automorphism group. A straightforward but lengthy computation using the algebra of  $\text{aut}_{\mathcal{H}(n)}^c$  shows that it does not admit an algebraic extension. Furthermore, all of the subgroups  $\mathcal{D}^+$  and  $\mathcal{H}(n)$  are simply connected. Only  $Sp(2n)$  has a nontrivial global topology and its fundamental homotopy group is  $\mathbb{Z}$ . Therefore the fully centrally extended connected component of the automorphism group is

$$\mathcal{Aut}_{\mathcal{H}(n)}^c \equiv \overline{\text{aut}_{\mathcal{H}(n)}^c} \simeq \mathcal{D}^+ \otimes_s \overline{Sp}(2n) \otimes_s \mathcal{H}(n). \quad (73)$$

The semidirect product  $\mathcal{H}\overline{Sp}(2n)$  of the cover of the symplectic group with the Weyl-Heisenberg group is defined using the homomorphism  $\pi : \overline{Sp}(2n) \rightarrow Sp(2n)$  with  $\ker \pi \simeq \mathbb{Z}$ . For  $\Sigma \in \overline{Sp}(2n)$ , the homomorphism  $\pi$  appears in the definition of the semidirect product is

$$\begin{aligned} \Omega(\epsilon'', \delta'', \Sigma'', z'', \iota'') &= \Omega(\epsilon', \delta', \Sigma', z', \iota') \Omega(\epsilon, \delta, \Sigma, z, \iota) \\ &= \Omega(\epsilon' \epsilon, \delta' \delta, \Sigma' \Sigma, z' + \epsilon' \delta' \Sigma' z, \iota' + \delta'^2 \iota - \frac{1}{2} \epsilon' \delta' z''^\dagger \zeta \pi(\Sigma') z), \end{aligned} \quad (74)$$

$$\Omega(\epsilon, \delta, \Sigma, z, \iota)^{-1} = \Omega(\epsilon, \delta^{-1}, \Sigma^{-1}, -\epsilon \delta^{-1} \pi(\Sigma)^{-1} z, -\delta^{-2} \iota). \quad (75)$$

where in this expression,  $\epsilon = 1$ .

As noted above, the central extension for a group that is not connected group is not necessarily unique and so we have to be careful before allowing  $\epsilon$  taking values of  $\epsilon \in \{\pm 1\} \simeq \mathbb{Z}_2$ . The central extension for a group that is not connected may be defined by requiring exact sequences both for the cover of the group and the homomorphisms onto the discrete group for the

components. For the  $\mathcal{A}ut_{\mathcal{H}(n)}$ , these sequences are [4]

$$\begin{array}{ccccccc}
& & e & & e & & e \\
& & \downarrow & & \downarrow & & \downarrow \\
e & \rightarrow & \mathbb{Z} & \rightarrow & \mathcal{A}ut_{\mathcal{H}(n)}^c & \rightarrow & \text{aut}_{\mathcal{H}(n)}^c \rightarrow e \\
& & \downarrow & & \downarrow & & \downarrow \\
e & \rightarrow & \mathbb{D} & \rightarrow & \mathcal{A}ut_{\mathcal{H}(n)} & \rightarrow & \text{aut}_{\mathcal{H}(n)} \rightarrow e \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \mathbb{Z}_2 & & \mathbb{Z}_2 & & \mathbb{Z}_2 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & e & & e & & e
\end{array} \tag{76}$$

A solution is  $\mathbb{D} \simeq \mathbb{Z}_2 \otimes \mathbb{Z} \simeq \mathbb{Z}$ . Therefore,

$$\mathcal{A}ut_{\mathcal{H}(n)} \simeq \mathbb{Z}_2 \otimes_s \mathcal{A}ut_{\mathcal{H}(n)}^c, \tag{77}$$

and the above group product expressions (61-62) are valid with  $\epsilon = \pm 1$ .

This completes the characterization of the automorphism group that we asserted in the introduction in equation (18).

### 3.0.1. Homomorphisms

Representations are homomorphisms of a group  $\mathcal{G}$ . If the homomorphism is an isomorphism, then the representation is said to be faithful and otherwise it is degenerate. Theorem 3 establishes that degenerate representations are faithful representations of groups homomorphic to  $\mathcal{G}$ . The homomorphisms can be characterized by the normal subgroups that are the kernel of the homomorphism.

First we consider the subgroup  $\mathcal{H}\overline{Sp}(2n)$  that we have noted in (20) is the central extension of  $ISp(2n)$  with center

$$\mathcal{Z} = \mathbb{Z} \otimes \mathcal{A}(1) \tag{78}$$

where  $\mathbb{Z}$  is the center of  $\overline{Sp}(2n)$  and  $\mathcal{A}(1)$  is the center of  $\mathcal{H}(n)$  (34). The double cover of  $Sp(2n)$  is the metaplectic group  $\mathcal{M}p(2n)$ . As  $\mathbb{Z}_2$  is a normal subgroup of  $\mathbb{Z}$ , that there is also a homomorphism from the cover of the symplectic group to the metaplectic group

$$\pi : \overline{Sp}(2n) \rightarrow \mathcal{M}p(2n), \quad \ker(\pi) \simeq \mathbb{Z}/\mathbb{Z}_2. \tag{79}$$

This gives the sequence of homomorphic groups where the homomorphisms have kernels that are subgroups of the center  $\mathcal{Z}$ .

$$\begin{array}{ccccccc}
\mathcal{H}\overline{Sp}(2n) & \rightarrow & \mathcal{H}\mathcal{M}p(2n) & \rightarrow & \mathcal{H}Sp(2n) & & \\
& \searrow & & \searrow & & \searrow & \\
& & \overline{ISp}(2n) & \rightarrow & I\mathcal{M}p(2n) & \rightarrow & ISp(2n).
\end{array} \tag{80}$$

The group  $ISp(2n)$  that has a trivial center terminates the sequence. It is the maximal *classical* symmetry group. The projective representations of any of the groups in this sequence is equivalent to the unitary representations of the  $\mathcal{H}\overline{Sp}(2n)$ .

Now, as  $\overline{\mathcal{HSp}}(2n)$  is a normal subgroup of  $\mathcal{Aut}_{\mathcal{H}(n)}$ , the kernels of the maps in the above are normal subgroups. They give rise to the homomorphism series of the automorphism group that have abelian kernels.

$$\begin{array}{ccccc} \mathcal{Aut}_{\mathcal{H}(n)} & \rightarrow & \mathcal{D} \otimes_s \mathcal{HMp}(2n) & \rightarrow & \mathcal{D} \otimes_s \mathcal{HSp}(2n) \\ & \searrow & & \searrow & \searrow \\ & & \overline{\mathcal{DSp}}(2n) & \rightarrow & \mathcal{DMp}(2n) & \rightarrow & \mathcal{DSp}(2n). \end{array} \quad (81)$$

In addition to the homomorphisms that have abelian kernels, we have the additional homomorphisms

$$\pi : \mathcal{Aut}_{\mathcal{H}(n)} \rightarrow \mathcal{G}, \ker(\pi) = \mathcal{K}, \quad (82)$$

with

$$\begin{array}{ll} \mathcal{K} & \mathcal{G} \\ \mathcal{H}(n) & \mathcal{D} \otimes \overline{\mathcal{Sp}}(2n) \\ \mathbb{Z}/\mathbb{Z}_2 \otimes \mathcal{H}(n) & \mathcal{D} \otimes \mathcal{Mp}(2n) \\ \mathbb{Z} \otimes \mathcal{H}(n) & \mathcal{D} \otimes \mathcal{Sp}(2n) \\ \mathcal{HSp}(2n) & \mathbb{Z} \otimes \mathcal{D} \\ \mathcal{HMp}(2n) & \mathbb{Z}_2 \otimes \mathcal{D} \\ \overline{\mathcal{HSp}}(2n) & \mathcal{D} \end{array} \quad (83)$$

### 3.0.2. Symplectic group factorization

The defining condition for the real symplectic group  $\mathcal{Sp}(2n)$  is

$$\Sigma^t \zeta \Sigma = \zeta \quad (84)$$

where  $\zeta$  is the symplectic matrix defined in (48). Matrix realizations of elements of the real symplectic group may be written as

$$\Sigma = \begin{pmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_3 & \Sigma_4 \end{pmatrix} \quad (85)$$

where  $\Sigma_a$ ,  $a = 1, \dots, 4$  are  $n \times n$  submatrices. The symplectic condition (84) immediately results in the relations

$$\begin{aligned} \Sigma_1^t \Sigma_4 - \Sigma_3^t \Sigma_2 &= 1_n, \\ \Sigma_1^t \Sigma_3 &= (\Sigma_1^t \Sigma_3)^t, \\ \Sigma_2^t \Sigma_4 &= (\Sigma_2^t \Sigma_4)^t. \end{aligned} \quad (86)$$

A matrix realization of a Lie group is a coordinate system. As  $\text{Det}(\Sigma) = 1$ , it follows that the determinate of at least one of the  $\Sigma_a$ ,  $a = 1, \dots, 4$ , must be nonzero. These correspond to different coordinate patches for the manifold underlying the symplectic group. Assume  $\text{Det}(\Sigma_1) \neq 0$ . Then [5],

$$\Sigma(\alpha, \beta, \gamma) = \begin{pmatrix} 1_n & 0 \\ \gamma & 1_n \end{pmatrix} \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha^t \end{pmatrix} \begin{pmatrix} 1_n & \beta \\ 0 & 1_n \end{pmatrix}, \quad (87)$$

where we define

$$\alpha = (\Sigma_1)^{-1}, \beta = (\Sigma_1)^{-1} \Sigma_2, \gamma = \Sigma_3 (\Sigma_1)^{-1}. \quad (88)$$

It follows from (86) that  $\beta = \beta^t$  and  $\gamma = \gamma^t$ . The matrix realizations of elements of the symplectic group factor as

$$\Sigma(\alpha, \beta, \gamma) = \Sigma^-(\gamma) \Sigma^\circ(\alpha) \Sigma^+(\beta) \quad (89)$$

where

$$\begin{aligned}\Sigma^\circ(\alpha) &\equiv \Sigma(\alpha, 1_n, 1_n) \in \mathcal{U}(n), \\ \Sigma^+(\beta) &\equiv \Sigma(1_n, \beta, 1_n) \in \mathcal{A}(m), \\ \Sigma^-(\gamma) &\equiv \Sigma(1_n, 1_n, \gamma) \in \mathcal{A}(m).\end{aligned}\tag{90}$$

and  $m = \frac{n(n-1)}{2}$ . Furthermore, note that

$$\zeta \Sigma^-(\gamma) \zeta^{-1} = \Sigma^+(-\gamma).\tag{91}$$

A similar argument applies if we instead assume  $\text{Det}(\Sigma_4) \neq 0$ . Both of these coordinate patches contain the identity,  $1_{2n}$  but neither contains the element  $\zeta$ . These require us to consider the case with either  $\Sigma_2$  or  $\Sigma_3$  to be assumed to be nonsingular. In this case, define

$$\tilde{\Sigma} = \Sigma \zeta^{-1} = \begin{pmatrix} \Sigma_2 & -\Sigma_1 \\ \Sigma_4 & -\Sigma_3 \end{pmatrix}.\tag{92}$$

The  $\tilde{\Sigma}$  also satisfy the symplectic condition as

$$\zeta = \Sigma^t \zeta \Sigma = \zeta^t \tilde{\Sigma}^t \zeta \tilde{\Sigma} \Rightarrow \tilde{\Sigma}^t \zeta \tilde{\Sigma} = \zeta.\tag{93}$$

This symplectic condition results in the identities

$$\begin{aligned}\Sigma_4^t \Sigma_1 - \Sigma_2^t \Sigma_3 &= 1_n, \\ \Sigma_2^t \tilde{\Sigma}_4 &= (\Sigma_2^t \Sigma_4)^t, \\ \Sigma_1^t \Sigma_3 &= (\Sigma_1^t \Sigma_3)^t.\end{aligned}\tag{94}$$

We can now assume  $\text{Det}(\Sigma_2) \neq 0$  and the analysis proceeds as before with

$$\alpha = (\Sigma_2)^{-1}, \beta = -(\Sigma_2)^{-1} \Sigma_1, \gamma = \Sigma_4 (\Sigma_2)^{-1},\tag{95}$$

In this case the factorization must include the symplectic matrix from (93)

$$\Sigma(\alpha, \beta, \gamma) = \Sigma^-(\gamma) \Sigma^\circ(\alpha) \Sigma^+(\beta) \zeta.\tag{96}$$

Finally a similar argument applies for the coordinate patch  $\text{Det}(\Sigma_3) \neq 0$ . Both of these coordinate patches contain the element  $\zeta$  but do not contain the identity

The expressions (89) and (96) can be combined into a single expression

$$\Sigma^\epsilon(\alpha, \beta, \gamma) = \Sigma^-(\gamma) \Sigma^\circ(\alpha) \Sigma^+(\beta) \zeta^\epsilon.\tag{97}$$

where  $\epsilon \in \{0, 1\}$ .

### 3.0.3. Lie Algebra

The Lie algebra of the automorphism group  $\mathcal{Aut}_{\mathcal{H}(n)}$  may be directly computed from its matrix realization. It is convenient to use a basis for the algebra of the symplectic group corresponding to the factorized form (89). Let the  $A_{i,j}$  be the generators of the unitary subgroup with elements  $\Sigma(\alpha) \in \mathcal{U}(n)$ , and  $B_{i,j}$  the generators of the abelian subgroup with elements  $\Sigma(\beta) \in \mathcal{A}(m)$  and  $C_{i,j}$  the generators of the abelian subgroup with elements  $\Sigma(\gamma) \in \mathcal{A}(m)$ . The abelian generators are symmetric,  $B_{i,j} = B_{j,i}$  and  $C_{i,j} = C_{j,i}$ . A general element is written as

$$Z = \delta Y + \alpha^{i,j} A_{i,j} + \beta^{i,j} B_{i,j} + \gamma^{i,j} C_{i,j} + p^i Q_i + q^i P_i + \iota I.\tag{98}$$

Straightforward computation shows that these generators of  $Sp(2n)$  satisfy the Lie algebra

$$\begin{aligned} [A_{i,j}, A_{k,l}] &= \delta_{jk}A_{i,l} - \delta_{il}A_{j,k}, \\ [A_{i,j}, B_{k,l}] &= \delta_{jk}B_{i,l} + \delta_{jl}B_{i,k}, \\ [A_{i,j}, C_{k,l}] &= -(\delta_{ik}C_{j,l} + \delta_{il}C_{k,j}), \\ [B_{i,j}, C_k] &= -(\delta_{i,k}A_{j,l} + \delta_{i,l}A_{j,k} + \delta_{j,k}A_{i,l} + \delta_{j,l}A_{i,k}). \end{aligned} \quad (99)$$

The nonzero commutators of the algebra of  $\overline{HSp}(2n)$  are the above relations for the symplectic generators together with the Weyl-Heisenberg generators are,

$$\begin{aligned} [A_{i,j}, Q_k] &= \delta_{jk}Q_i, \quad [F_{i,j}, Q_k] = \delta_{i,k}P_j + \delta_{j,k}P_i, \\ [A_{i,j}, P_k] &= -\delta_{i,k}P_j, \quad [B_{i,j}, P_k] = -\delta_{i,k}Q_j - \delta_{j,k}Q_i, \\ [P_i, Q_j] &= \delta_{i,j}I. \end{aligned} \quad (100)$$

Finally, the Lie algebra of the full  $\mathcal{Aut}_{\mathcal{H}(n)}$  group are the above relations together with the commutators of the generator  $Y$  of  $\mathcal{D}$ ,

$$[Q_k, Y] = Q_k, [P_k, Y] = P_k, [I, Y] = 2I. \quad (101)$$

All other commutators are zero. Note that  $Y$  does not commute with  $I$  and so it is not a central generator of the full group.

#### 4. Quantum symmetry: Projective representations

The projective representations of the automorphism group  $\mathcal{Aut}_{\mathcal{H}(n)}$  are ordinary unitary representations as the group is maximally centrally extended. The unitary irreducible representations of this group may be determined using the Mackey theorems for semidirect product groups.

The first step is to determine the unitary irreducible representations of the normal subgroup that is the Weyl-Heisenberg group. The unitary irreducible representations of the Weyl-Heisenberg group are very well known from the Stone-von Neumann theorem. This theorem however is not constructive. Rather it assumes that you have determined the representations in some unspecified manner and then theorem proves is that these representations are the complete set of unitary, irreducible representations for the Weyl-Heisenberg group.

The Mackey theorems are much more powerful in that they provide a method of constructing the unitary irreducible representations of a general semidirect product group, which of course includes the Weyl-Heisenberg group as a particular case. The power of the theorems is that they can also be used for other semidirect products including  $\mathcal{Aut}_{\mathcal{H}(n)}$ .

The first step in applying the Mackey Theorem 8 is to determine the unitary irreducible representations of the Weyl-Heisenberg normal subgroup. While these are well known, the method of constructing them using the Mackey theorems applied to the semidirect product of two abelian groups (10) does not appear to be as well known. We review this briefly in order to introduce the Mackey theorems and to show how the unitary irreducible representations of the automorphism group can be constructed completely from first principles.

##### 4.1. Unitary irreducible representations of the Weyl-Heisenberg group

The Mackey theorem for semidirect products with an abelian normal subgroup, Theorem 8 in the appendix, may now be applied [6], [7]. We choose the normal subgroup with elements

$\Upsilon(p, 0, \iota) \in \mathcal{A}(n+1)$ . The unitary irreducible representations  $\xi$  of the abelian normal subgroup are the phases acting on the Hilbert space  $\mathbf{H}^\xi = \mathbb{C}$

$$\xi(\Upsilon(p, 0, \iota))|\phi\rangle = e^{i(\hat{U}+p^i\hat{Q}_i)}|\phi\rangle = e^{i(\iota\lambda+p\cdot\alpha)}|\phi\rangle, \quad |\phi\rangle \in \mathbb{C}. \quad (102)$$

The hermitian representation of the algebra has the eigenvalues that are given by

$$\hat{Q}_i|\phi\rangle = \xi'(Q_i)|\phi\rangle = \alpha_i|\phi\rangle, \quad \hat{I}|\phi\rangle = \xi'(I)|\phi\rangle = \lambda|\phi\rangle, \quad (103)$$

where  $\alpha \in \mathbb{R}^n$  and  $\lambda \in \mathbb{R}$ . The characters  $\xi_{\alpha,\lambda}$  are parameterized by the eigenvalues  $\alpha, \lambda$  and the equivalence classes that are elements of the unitary dual,  $[\xi_{\alpha,\lambda}] \in \mathcal{U}_{\mathcal{A}(n+1)} \simeq \mathbb{R}^{n+1}$ . Each equivalence class has the single element  $[\xi_{\alpha,\lambda}] = \xi_{\alpha,\lambda}$ .

The action of the elements  $\Upsilon(0, q, 0) \in \mathcal{A}(n)$  of the homogeneous group on these representations is given by the dual automorphisms

$$\begin{aligned} (\hat{\mathcal{S}}_{\Upsilon(0,q,0)}\xi_{\alpha,\lambda})(\Upsilon(p, 0, \iota))|\phi\rangle &= \xi_{\alpha,\lambda}(\mathcal{S}_{\Upsilon(0,q,0)}\Upsilon(p, 0, \iota))|\phi\rangle \\ &= \xi_{\alpha-\lambda q,\lambda}(\Upsilon(p, 0, \iota))|\phi\rangle. \end{aligned} \quad (104)$$

In simplifying this expression, we have used (33) and (102). The little group is the set of  $\Upsilon(0, q, 0) \in \mathcal{K}^\circ$  that satisfy the fixed point equation (154),

$$\hat{\mathcal{S}}_{\Upsilon(0,q,0)}\xi_{\alpha,\lambda} = \xi_{\alpha-\lambda q,\lambda} = \xi_{\alpha,\lambda}. \quad (105)$$

The solution of the fixed point condition requires that  $\alpha - \lambda q \equiv \alpha$ . The  $\lambda = 0$  solution for which the little group is  $\mathcal{A}(n)$  is the degenerate case corresponding to the homomorphism  $\mathcal{H}(n) \rightarrow \mathcal{A}(2n)$  with kernel  $\mathcal{A}(1)$ . This is just the abelian group that is not considered further here. The faithful representation with  $\lambda \neq 0$  requires  $p = 0$ , and therefore has the trivial little group  $\mathcal{K}^\circ \simeq \mathbf{e} \simeq \{\Upsilon(0, 0, 0)\}$ . The stabilizer is  $\mathcal{G}^\circ \simeq \mathcal{A}(n+1)$ . The orbits are

$$\mathbb{O}_\lambda = \{\hat{\mathcal{S}}_{\Upsilon(0,q,0)}[\xi_{\alpha,\lambda}] | q \in \mathbb{R}^n\} = \{\xi_{q,\lambda} | q \in \mathbb{R}^n\}, \quad \lambda \in \mathbb{R} \setminus \{0\}. \quad (106)$$

All representations in the orbit are equivalent for the determination of the semidirect product unitary irreducible representations. A convenient representative of the equivalence class is  $\xi_{0,\lambda}$ . The unitary representations  $\sigma$  of the trivial little group are trivial and therefore the representations of the stabilizer are just  $\varrho^\circ = \xi_{0,\lambda}$ . The Hilbert space  $\mathbf{H}^\sigma$  is also trivial and therefore the Hilbert space of the stabilizer is  $\mathbf{H}^{\varrho^\circ} = \mathbf{H}^\sigma \otimes \mathbf{H}^\xi \simeq \mathbb{C}$ .

#### 4.1.1. Mackey induction

The final step is to apply the Mackey induction theorem to determine the faithful unitary irreducible representations of the full  $\mathcal{H}(n)$  group. The induction requires the definition of the symmetric space

$$\mathbb{K} = \mathcal{G}/\mathcal{G}^\circ = \mathcal{H}(n)/\mathcal{A}(n+1) \simeq \mathcal{A}(n) \simeq \mathbb{R}^n, \quad (107)$$

with the natural projection  $\pi$  and a section  $\Theta$

$$\begin{aligned} \pi : \mathcal{H}(n) &\rightarrow \mathbb{K} : \Upsilon(p, q, \iota) \mapsto k_q, \\ \Theta : \mathbb{K} &\rightarrow \mathcal{H}(n) : k_q \mapsto \Theta(k_q) = \Upsilon(0, q, 0). \end{aligned} \quad (108)$$

These satisfy  $\pi(\Theta(a_q)) = a_q$  and so  $\pi \circ \Theta = \text{Id}_{\mathbb{K}}$  as required. Using (2), an element of the Weyl-Heisenberg group  $\mathcal{H}(n)$  can be written as,

$$\Upsilon(p, q, \iota) = \Upsilon(0, q, 0)\Upsilon(p, 0, \iota + \frac{1}{2}p \cdot q). \quad (109)$$

The cosets are therefore defined by

$$\begin{aligned} \mathbf{k}_q &= \{ \Upsilon(0, q, 0)\Upsilon(p, 0, \iota + \frac{1}{2}p \cdot q) | p \in \mathbb{R}^n, \iota \in \mathbb{R} \} \\ &= \{ \Upsilon(0, q, 0)\mathcal{A}(n+1) \} \end{aligned} \quad (110)$$

Note that

$$\Upsilon(p, q, \iota)\mathbf{k}_x = \mathbf{k}_{x+q}, \quad x \in \mathbb{R}^n. \quad (111)$$

The Mackey induced representation theorem can now be applied straightforwardly. First, the Hilbert space is

$$\mathbf{H}^q = L^2(\mathbb{K}, \mathbf{H}^{q^\circ}) \simeq L^2(\mathbb{R}^n, \mathbb{C}). \quad (112)$$

Next the Mackey induction Theorem 6 yields

$$\psi'(\mathbf{k}_x) = (\varrho(\Upsilon(p, q, \iota))\psi) \left( \Upsilon(p, q, \iota)^{-1}\mathbf{k}_x \right) = \varrho^\circ(\Upsilon(a^\circ, 0, \iota^\circ))\psi(\mathbf{k}_{x-q}) \quad (113)$$

Using the Weyl-Heisenberg group product (2),

$$\begin{aligned} \Upsilon(p^\circ, q^\circ, \iota^\circ) &= \Theta(\mathbf{k}_x)^{-1}\Upsilon(p, q, \iota)\Theta(\Upsilon(p, q, \iota)^{-1}\mathbf{k}_x) \\ &= \Upsilon(0, -x, 0)\Upsilon(p, q, \iota)\Upsilon(0, x - q, 0) \\ &= \Upsilon(p, 0, \iota + p \cdot (x - \frac{1}{2}q)). \end{aligned} \quad (114)$$

We lighten notation using the isomorphism  $\mathbf{k}_x \mapsto x$ . The induced representation theorem then yields

$$\begin{aligned} \psi'(x) &= \xi_{0,\lambda}(\Upsilon(p, 0, \iota + x \cdot p - \frac{1}{2}p \cdot q)\psi(x - q)) \\ &= e^{i\lambda(\iota + x \cdot p - \frac{1}{2}p \cdot q)}\psi(x - q). \end{aligned} \quad (115)$$

Using Taylor expansion, we can write

$$\psi(x - q) = e^{-q^i \frac{\partial}{\partial x^i}} \psi(x). \quad (116)$$

The Baker Campbell-Hausdorff formula [20] enables us to combine the exponentials

$$\psi'(x) = e^{i(\lambda\iota + \lambda p^i x_i + q^i i \frac{\partial}{\partial x^i})}\psi(x) = e^{i(\hat{U} + p^i \hat{Q}_i + q^i \hat{P}_i)}\psi(x). \quad (117)$$

The representation of the algebra is therefore

$$\hat{U}\psi(x) = \lambda\psi(x), \quad \hat{Q}_i\psi(x) = \lambda x_i\psi(x), \quad \hat{P}_i\psi(x) = i \frac{\partial}{\partial x^i}\psi(x), \quad (118)$$

that satisfies the Heisenberg commutation relations (1).

This analysis can also be carried out choosing  $\Upsilon(0, q, \iota) \in \mathcal{A}(n+1)$  to be the elements of the normal subgroup and this yields the representation with  $\hat{P}_i$  diagonal.

#### 4.2. Unitary irreducible representations of $\overline{\mathcal{HSp}}(2n)$

We consider next the unitary irreducible representations of the  $\overline{\mathcal{HSp}}(2n)$  normal subgroup of  $\mathcal{Aut}_{\mathcal{H}(n)}$

$$\overline{\mathcal{HSp}}(2n) \simeq Sp(2n) \otimes_s \mathcal{H}(n). \quad (119)$$

As  $\overline{\mathcal{HSp}}(2n)$  is the central extension of  $ISp(2n)$ , the projective representations of  $ISp(2n)$  are equivalent to the ordinary unitary representations of  $\overline{\mathcal{HSp}}(2n)$ .

The unitary irreducible representations of  $\overline{\mathcal{HSp}}(2n)$  may be determined using Mackey Theorem 7 for the nonabelian normal subgroup case. The faithful unitary representations of the Weyl-Heisenberg group are given in the previous section (115). The next step in applying the Mackey's theorem is to determine the  $\rho$  representation of the stabilizer  $\mathcal{G}^\circ \subset \overline{\mathcal{HSp}}(2n)$ .

##### 4.2.1. Stabilizer and $\rho$ representation

The representation  $\rho$  of the stabilizer  $\mathcal{G}^\circ$  acts on the Hilbert space  $\mathbf{H}^\xi$  and therefore the hermitian representations  $\rho'$  of the algebra of the stabilizer must be realized in the enveloping algebra of the Weyl-Heisenberg group. The  $\rho$  representation restricted to the Weyl-Heisenberg group are given by  $\rho|_{\mathcal{H}(n)} = \xi$  where  $\xi$  are the unitary irreducible representations of the Weyl Heisenberg group. The faithful representations  $\xi$  are given in (115).

The unitary representation  $\rho$  acts on  $\mathbf{H}^\xi \simeq L^2(\mathbb{R}^n, \mathbb{C})$  such that

$$\rho(\Omega^\circ) \xi(\Upsilon(z, \iota)) \rho(\Omega^\circ)^{-1} = \xi(\zeta_{\Omega^\circ} \Upsilon(z, \iota)), \quad \Omega^\circ \in \mathcal{G}^\circ. \quad (120)$$

The representation  $\rho$  factors into

$$\rho(\Omega^\circ(\delta, \Sigma, w, r)) = \xi(\Upsilon(w, r)) \rho(\Sigma), \quad (121)$$

where again for notational brevity  $\Sigma \equiv \Omega(1, \Sigma, 0, 0)$ .

We already have characterized the inner automorphisms. The automorphisms corresponding factor as

$$\begin{aligned} \xi(\Upsilon(w, r)) \xi(\Upsilon(z, \iota)) \xi(\Upsilon(w, r))^{-1} &= \xi(\zeta_{\Upsilon(w, r)} \Upsilon(z, \iota)), \\ \rho(\Sigma) \xi(\Upsilon(z, \iota)) \rho(\Sigma)^{-1} &= \xi(\zeta_{\Omega(\Sigma)} \Upsilon(z, \iota)) = \xi(\Upsilon(\pi(\Sigma)z, \iota)). \end{aligned} \quad (122)$$

where  $\Sigma \in \overline{Sp}(2n)$  and  $\pi : \overline{Sp}(2n) \rightarrow Sp(2n)$ .

The inner automorphisms are already characterized as we know the unitary irreducible representations  $\xi$ . Consider next the representation  $\rho(\Sigma)$  of the symplectic group  $\overline{Sp}(2n)$ . The hermitian representation of the symplectic generators is

$$\begin{aligned} \hat{A}_{i,j} &= \rho'(A_{i,j}) = \lambda \hat{Q}_i \hat{P}_j, \\ \hat{B}_{i,j} &= \rho'(B_{i,j}) = \lambda \hat{Q}_i \hat{Q}_j, \\ \hat{C}_{i,j} &= \rho'(C_{i,j}) = \lambda \hat{P}_i \hat{P}_j. \end{aligned} \quad (123)$$

Clearly  $\hat{B}_{i,j} = \hat{B}_{j,i}$  and  $\hat{C}_{i,j} = \hat{C}_{j,i}$ . Then, using the Heisenberg commutation relations (1), this defines a hermitian realization of the Lie algebra of the automorphism group acting on the Hilbert space  $\mathbf{H}^\xi \simeq L^2(\mathbb{R}^n, \mathbb{C})$ .

$$\begin{aligned} [\hat{A}_{i,j}, \hat{A}_{k,l}] &= i(\delta_{j,k} \hat{A}_{i,l} - \delta_{i,l} \hat{A}_{j,k}), \\ [\hat{A}_{i,j}, \hat{B}_{k,l}] &= i(\delta_{j,k} \hat{B}_{i,l} + \delta_{j,l} \hat{B}_{i,k}), \\ [\hat{A}_{i,j}, \hat{C}_{k,l}] &= -i(\delta_{i,k} \hat{C}_{j,l} + \delta_{i,l} \hat{C}_{k,j}), \\ [\hat{B}_{i,j}, \hat{C}_{k,l}] &= -i(\delta_{i,k} \hat{A}_{j,l} + \delta_{i,l} \hat{A}_{j,k} + \delta_{j,k} \hat{A}_{i,l} + \delta_{j,l} \hat{A}_{i,k}). \end{aligned} \quad (124)$$



Therefore, there exists a  $\rho'$  representation for the entire algebra of  $\overline{\mathcal{HSp}}(2n)$  and therefore the stabilizer is the group itself,  $\mathcal{G}^\circ \simeq \overline{\mathcal{HSp}}(2n)$ . This explicate construction of the algebra shows that the representation  $\rho(\Sigma)$  exists. Consequently, the Mackey induction theorem is not required.

The  $\rho(\Sigma)$  representation is precisely (up to an overall phase) the metaplectic representation originally studied by Weyl [5], [3]. We can construct this explicitly using the factorization of the symplectic group (97). We can consider each of the factors separately as

$$\rho(\Sigma(\epsilon, \alpha, \beta, \gamma)) = \rho(\Sigma^-(\gamma))\rho(\Sigma^\circ(\alpha))\rho(\Sigma^+(\beta))\rho(\zeta^\epsilon), \quad (125)$$

and each of these factors can be applied separately to determine the  $\rho$  representation. The unitary representations of  $\Sigma(\beta) \in \mathcal{A}(m)$ ,  $m = \frac{n(n+1)}{2}$  in a basis with  $\hat{Q}_i$  diagonal are

$$\rho(\Sigma^+(\beta))|\psi_\lambda(x)\rangle = e^{i\alpha^{ij}\hat{B}_{i,j}}|\psi_\lambda(x)\rangle = e^{\frac{i}{\lambda}\beta^{ij}x_i x_j}|\psi_\lambda(x)\rangle. \quad (126)$$

The representations of the elements of the unitary group  $\Sigma(\alpha) \in \mathcal{U}(n)$  are

$$\rho(\Sigma^\circ(\alpha))|\psi_\lambda(x)\rangle = |\det A|^{-\frac{1}{2}}|\psi_\lambda(A^{-1}x)\rangle. \quad (127)$$

The symplectic matrix exchanges the  $p$  and  $q$  degrees of freedom,  $\zeta_\epsilon \Upsilon(p, q, \iota) = \Upsilon(q, -p, \iota)$ . As is well known, the unitary representation of this is the Fourier transform,  $\rho(\zeta) = f$  where

$$\rho(\Upsilon(p, q, \iota))f|\psi_\lambda(x)\rangle = f\rho(\Upsilon(q, -p, \iota))|\psi_\lambda(x)\rangle, \quad (128)$$

where the Fourier transform is defined as usual by

$$\tilde{\psi}(y) = f\psi(x) = (2\pi i)^{-\frac{n}{2}} \int e^{-ix \cdot y} \psi(x) d^n x, \quad (129)$$

and where

$$\hat{Q}_i|\psi_\lambda(x)\rangle = \lambda x_i |\psi_\lambda(x)\rangle, \quad \hat{P}_i|\tilde{\psi}_\lambda(y)\rangle = y_i |\tilde{\psi}_\lambda(y)\rangle. \quad (130)$$

Finally, the  $\rho(\Sigma^+(\beta))$  representation can be computed using (91) in a basis with  $\hat{Q}_i$  diagonal giving

$$\rho(\Sigma^-(\gamma))|\psi_\lambda(x)\rangle = f\rho(\Sigma^+(-\gamma))f^{-1}|\psi_\lambda(x)\rangle, \quad (131)$$

and the  $\rho(\Sigma^+(-\gamma))$  is given by (126). Putting all of these together gives the representation  $\rho(\Sigma)$  up to a phase. While one would expect the phase to be  $m \in \mathbb{Z}$  dependent, it actually only is two valued  $\pm 1 \in \mathbb{Z}_2$ . The unitary representations of the double cover metaplectic group  $\mathcal{Mp}(2n)$  are also a representation of  $\overline{\mathcal{Sp}}(2n)$  due to the homomorphism (79).

Of course, all of these calculations could also be done in a basis with  $\hat{P}_i$  diagonal.

As the stabilizer is the full group, Mackey induction is not required and the unitary irreducible representations  $\nu$  of  $\overline{\mathcal{HSp}}(2n)$  are given by

$$\nu(\Omega(1, \Sigma, z, \iota))|\psi(x)\rangle = \sigma(\Sigma) \otimes \xi(\Upsilon(z, \iota))\rho(\Sigma)|\psi(x)\rangle \quad (132)$$

where  $\sigma$  are ordinary unitary irreducible representations of  $\overline{\mathcal{Sp}}(2n)$ ,  $\rho$  are the metaplectic representation of  $\overline{\mathcal{Sp}}(2n)$  given above and  $\xi$  are the unitary irreducible representations of  $\mathcal{H}(n)$  given in Section 4.1.

The ordinary unitary representations of the symplectic group have been partially characterized [8, 9]. A complete set of unitary irreducible representations of the covering group  $\overline{\mathcal{Sp}}(2n)$  appears to be an open problem.

#### 4.3. Unitary irreducible representations of $\mathcal{A}ut_{\mathcal{H}(n)}^c$

The connected component of the automorphism group is a semidirect product of the form given in (73).  $\mathcal{H}\overline{Sp}(2n)$  is the normal subgroup and  $\mathcal{D}^+$  is the homogeneous group. Again, we apply the Mackey Theorem 7. We seek a unitary representation  $\rho$  of the stabilizer  $\mathcal{G}^\circ$  such that when it is restricted to  $\mathcal{H}\overline{Sp}(2n)$ , it is the  $\rho$  representation described in the previous section. Furthermore, as elements of  $\mathcal{D}^+$  commute with elements of  $\overline{Sp}(2n)$ , this must also be the case for the representation.

Using the same method as the previous section, we first look for a  $\rho'(Y)$  where  $Y$  is the generators of  $\mathcal{D}^+$  in the enveloping algebra of  $\mathcal{H}(n)$ . Consequently, a nontrivial unitary representation of  $\mathcal{D} \otimes \overline{Sp}(2n)$  acting on the Hilbert space  $\mathbf{H}^\xi$  does not exist.

Therefore, the stabilizer  $\mathcal{G}^\circ \simeq \mathcal{H}\overline{Sp}(2n)$  and the little group is  $\mathcal{K}^\circ \simeq \overline{Sp}(2n)$ . The final step is to use the Mackey Induction Theorem 6 to compute the full representation.

The homogeneous space is

$$\mathbb{K} \simeq \mathcal{A}ut_{\mathcal{H}(n)}/\mathcal{G}^\circ \simeq \mathcal{D} \otimes_s \mathcal{H}\overline{Sp}(2n)/\mathcal{H}\overline{Sp}(2n) \simeq \mathbb{R}^+. \quad (133)$$

The projection yields the cosets

$$\pi : \mathcal{A}ut_{\mathcal{H}(n)} \rightarrow \mathbb{K} : \Omega(\delta)\Omega(\Sigma) \mapsto k_\delta = \left\{ \Omega(\delta)\overline{Sp}(2n) \right\}. \quad (134)$$

and the section is

$$\Theta : \mathbb{K} \rightarrow \mathcal{D} \otimes_s \overline{Sp}(2n) : k_\delta \mapsto \Omega(\delta). \quad (135)$$

The action of a group element on a coset is

$$\Omega(\tilde{\delta}, \Sigma, z, \iota)k_{\tilde{\delta}} = k_{\delta\tilde{\delta}}. \quad (136)$$

The element of the stabilizer is

$$\begin{aligned} \Omega^\circ &= \Theta(k_{\tilde{\delta}})^{-1} \Omega(\delta, \Sigma, \zeta, \iota) \Theta(\Omega(\delta, \Sigma, 0, 0)^{-1} k_{\tilde{\delta}}) \\ &= \Omega(\tilde{\delta}^{-1}) \Omega(\delta, \Sigma, z, \iota) \Omega(\delta^{-1} \tilde{\delta}) = \Omega(1, \Sigma, \delta^{-1} z, \delta^{-2} \iota). \end{aligned} \quad (137)$$

Therefore, the faithful unitary irreducible representations of the connected component of the automorphism group,  $\mathcal{A}ut_{\mathcal{H}(n)}^c$ , are

$$v(\Omega(\delta, \Sigma, z, \iota))|\psi(x)\rangle = \sigma(\Sigma) \otimes \xi(\Upsilon(\delta^{-1} z, \delta^{-2} \iota)) \rho(\Sigma) |\psi(\delta^{-1} x)\rangle, \quad (138)$$

where the terms in the expression are defined as in (132).

The action on the central generator of the Weyl-Heisenberg group is

$$v(\Omega(\delta, \Sigma, z, \iota)) \hat{I} v(\Omega(\delta, \Sigma, z, \iota))^{-1} = \delta^2 \hat{I}. \quad (139)$$

#### 4.4. Unitary irreducible representations of $\mathcal{A}ut_{\mathcal{H}(n)}$

The central extension of a group that is not connected is not necessarily unique. However, in this case, the central extension is unique and is defined by (77). The projective representation of elements of a group that is not connected may be anti-unitary and anti-linear.

The representations of the connected component  $\mathcal{Aut}_{\mathcal{H}(n)}^c$  are unitary and linear as described in the next section. Elements with  $\epsilon = -1$  are in the component not connected to the identity. Define  $\Omega(\epsilon) \equiv \Omega(\epsilon, 1, 1_{2n}, 0, 0)$  and the semidirect property (77) enables us to write

$$\Omega(\epsilon, \delta, \Sigma, z, \iota) = \Omega(\delta, \Sigma, z, \iota) \Omega(\epsilon). \quad (140)$$

For,  $\epsilon = 1$ ,  $\Omega(\epsilon)$  is the identity. For  $\epsilon = -1$ , define

$$B \equiv \Omega(\epsilon), \epsilon = -1. \quad (141)$$

The action of  $B$  on the central generator of the Weyl-Heisenberg group may have either sign

$$\nu(B) \hat{I} \nu(B)^{-1} = \pm \hat{I}. \quad (142)$$

If the sign is positive, then the elements in the component with  $\epsilon = -1$  are anti-unitary and anti-linear. If the sign is negative, they are unitary and linear. The choice is dictated by physical rather than mathematical considerations. Certainly physically, we would expect the sign to be positive.

## 5. Summary

We have determined the projective representations of the automorphism group of the Weyl-Heisenberg group. This is the maximal symmetry whose projective representations transform physical states such that the Heisenberg commutation relations are valid in all of the transformed states.

A symmetry is physical if the group preserves the Heisenberg commutation relations in this manner. This maximal symmetry is important as any symmetry that is physical must be a subgroup of it and its projective representations are contained in the maximal representations that we have determined in the paper.

This theory is formulated with degrees of freedom that are physically interpreted as phase space. Nothing in the above analysis favors the position degree of freedom as being more fundamental than the momentum degree of freedom or vice versa. The representations can always be formulated with either  $\hat{P}_i$  or  $\hat{Q}_i$  diagonal.

## 6. Appendix A: Key Theorems

In this appendix we review a set of definitions and theorems that are fundamental for the application of symmetry groups in quantum mechanics. We state the theorems only and refer the reader to the cited literature for full proofs. Our notation for a semidirect product is  $\mathcal{G} \simeq \mathcal{K} \otimes_s \mathcal{N}$  where  $\mathcal{N}$  is the normal subgroup and  $\mathcal{K}$  is the homogeneous subgroup such that  $\mathcal{K} \cap \mathcal{N} = \mathbf{e}$  and  $\mathcal{G} \simeq \mathcal{N}\mathcal{K}$ . A semidirect product is right associative in the sense that  $\mathcal{D} \simeq (\mathcal{A} \otimes_s \mathcal{B}) \otimes_s \mathcal{C}$  implies that  $\mathcal{D} \simeq \mathcal{A} \otimes_s (\mathcal{B} \otimes_s \mathcal{C})$  and so brackets can be removed. However  $\mathcal{D} \simeq \mathcal{A} \otimes_s (\mathcal{B} \otimes_s \mathcal{C})$  does not necessarily imply  $\mathcal{D} \simeq (\mathcal{A} \otimes_s \mathcal{B}) \otimes_s \mathcal{C}$  as  $\mathcal{B}$  is not necessarily a normal subgroup of  $\mathcal{A}$ .

**Definition 1.** An algebraic central extension of a Lie algebra  $\mathfrak{g}$  is the Lie algebra  $\check{\mathfrak{g}}$  that satisfies the following short exact sequence where  $\mathfrak{z}$  is the maximal abelian algebra that is central in  $\check{\mathfrak{g}}$ ,

$$\mathbf{0} \rightarrow \mathcal{Z} \rightarrow \check{\mathcal{G}} \rightarrow \mathfrak{g} \rightarrow \mathbf{0}. \quad (143)$$

where  $\mathbf{0}$  is the trivial algebra. Suppose  $\{X_a\}$  is a basis of the Lie algebra  $\mathfrak{g}$  with commutation relations  $[X_a, X_b] = c_{a,b}^c X_c$ ,  $a, b = 1, \dots, r$ . Then an algebraic central extension is a maximal set of central abelian generators  $\{A_\alpha\}$ , where  $\alpha, \beta, \dots = 1, \dots, m$ , such that

$$[A_\alpha, A_\beta] = 0, \quad [X_a, A_\alpha] = 0, \quad [X_a, X_b] = c_{a,b}^c X_c + c_{a,b}^\alpha A_\alpha. \quad (144)$$

The basis  $\{X_a, A_\alpha\}$  of the centrally extended Lie algebra must also satisfy the Jacobi identities. The Jacobi identities constrain the admissible central extensions of the algebra. The choice  $X_a \mapsto X_a + A_a$  will always satisfy these relations and this trivial case is excluded. The algebra  $\check{\mathcal{G}}$  constructed in this manner is equivalent to the central extension of  $\mathfrak{g}$  given in Definition 1.

**Definition 2.** The central extension of a connected Lie group  $\mathcal{G}$  is the Lie group  $\check{\mathcal{G}}$  that satisfies the following short exact sequence where  $\mathcal{Z}$  is a maximal abelian group that is central in  $\check{\mathcal{G}}$

$$\mathbf{e} \rightarrow \mathcal{Z} \rightarrow \check{\mathcal{G}} \xrightarrow{\pi} \mathcal{G} \rightarrow \mathbf{e}. \quad (145)$$

The abelian group  $\mathcal{Z}$  may always be written as the direct product  $\mathcal{Z} \simeq \mathcal{A}(m) \otimes \mathbb{A}$  of a connected continuous abelian Lie group  $\mathcal{A}(m) \simeq (\mathbb{R}^m, +)$  and a discrete abelian group  $\mathbb{A}$  that may have a finite or countable dimension [10].

The exact sequence may be decomposed into an exact sequence for the *topological* central extension and the *algebraic* central extension,

$$\mathbf{e} \rightarrow \mathbb{A} \rightarrow \overline{\mathcal{G}} \xrightarrow{\pi^\circ} \check{\mathcal{G}} \rightarrow \mathbf{e}, \quad \mathbf{e} \rightarrow \mathcal{A}(m) \rightarrow \check{\mathcal{G}} \xrightarrow{\tilde{\pi}} \overline{\mathcal{G}} \rightarrow \mathbf{e}. \quad (146)$$

where  $\pi = \pi^\circ \circ \tilde{\pi}$ . The first exact sequence defines the universal cover where  $\mathbb{A} \simeq \ker \pi^\circ$  is the fundamental homotopy group. All of the groups in the second sequence are simply connected and therefore may be defined by the exponential map of the central extension of the algebra given by Definition 1. In other words, the full central extension may be computed by determining the universal covering group of the algebraic central extension.

**Definition 3.** A ray  $\Psi$  is the equivalence class of states  $|\psi\rangle$  that are elements of a Hilbert space  $\mathbf{H}$  up to a phase,

$$\Psi = \{e^{i\omega} |\psi\rangle \mid \omega \in \mathbb{R}\}, \quad |\psi\rangle \in H. \quad (147)$$

Note that the physical probabilities that are the square of the modulus depend only on the ray

$$|(\Psi_\beta, \Psi_\alpha)|^2 = |\langle \psi_\beta | \psi_\alpha \rangle|^2$$

for all  $|\psi_\gamma\rangle \in \Psi$ . For this reason, physical states in quantum mechanics are defined to be rays rather than states in the Hilbert space

**Definition 4.** A projective representation  $\varrho$  of a symmetry group  $\mathcal{G}$  is the maximal representation such that for  $|\tilde{\psi}_\gamma\rangle = \varrho(g)|\psi_\gamma\rangle$ , the modulus is invariant  $|\langle \tilde{\psi}_\beta | \tilde{\psi}_\alpha \rangle|^2 = |\langle \psi_\beta | \psi_\alpha \rangle|^2$  for all  $|\psi_\gamma\rangle, |\tilde{\psi}_\gamma\rangle \in \Psi$ .

**Theorem 1. (Wigner, Weinberg):** Any projective representation of a Lie symmetry group  $\mathcal{G}$  on a separable Hilbert space is equivalent to a representation that is either linear and unitary or anti-linear and anti-unitary. Furthermore, if  $\mathcal{G}$  is connected, the projective representations are equivalent to a representation that is linear and unitary [1],[11].

This is the generalization of the well known theorem that the ordinary representation of any compact group is equivalent to a representation that is unitary. For a projective representation, the phase degrees of freedom of the central extension enables the equivalent linear unitary or anti-linear anti-unitary representation to be constructed for this much more general class of Lie groups that admit representations on separable Hilbert spaces. (A proof of the theorem is given in Appendix A of Chapter 2 of [1]). The set of groups that this theorem applies to include all the groups that are studied in this paper.

**Theorem 2. (Bargmann, Mackey)** *The projective representations of a connected Lie group  $\mathcal{G}$  are equivalent to the ordinary unitary representations of its central extension  $\tilde{\mathcal{G}}$  [10, 12].*

Theorem 1 states that all projective representations are equivalent to a projective representation that is unitary. A phase is the unitary representation of a central abelian subgroup. Therefore, the maximal representation is given in terms of the central extension of the group. Appendix A shows how this definition is equivalent to the formulation of a projective representation as an ordinary unitary representation that is defined up to a phase,  $\varrho(\gamma')\varrho(\gamma) = e^{i\theta}\varrho(\gamma'\gamma)$  [12],[13].

**Theorem 3.** *Let  $\mathcal{G}, \mathcal{H}$  be Lie groups and  $\pi : \mathcal{G} \rightarrow \mathcal{H}$  be a homomorphism. Then, for every unitary representation  $\tilde{\varrho}$  of  $\mathcal{H}$  there exists a degenerate unitary representation  $\varrho$  of  $\mathcal{G}$  defined by  $\varrho = \tilde{\varrho} \circ \pi$ . Conversely, for every degenerate unitary representation of a Lie group  $\mathcal{G}$  there exists a Lie subgroup  $\mathcal{H}$  and a homomorphism  $\pi : \mathcal{G} \rightarrow \mathcal{H}$  where  $\ker(\pi) \neq \mathbf{e}$  such that  $\varrho = \tilde{\varrho} \circ \pi$  where  $\tilde{\varrho}$  is a unitary representation of  $\mathcal{H}$ .*

Noting that a representation is a homomorphism, This theorem follows straightforwardly from the properties of homomorphisms. As a consequence, the set of degenerate representations of a group is characterized by its set of normal subgroups. A *faithful* representation is the case that the representation is an isomorphism.

**Theorem 4. (Levi)** *Any simply connected Lie group is equivalent to the semidirect product of a semisimple group and a maximal solvable normal subgroup [13]*

As the central extension of any connected group is simply connected, the problem of computing the projective representations of a group always can be reduced to computing the unitary irreducible representations of a semidirect product group with a semisimple homogeneous group and a solvable normal subgroup. The unitary irreducible representations of the semisimple groups are known and the solvable groups that we are interested in turn out to be the semidirect product of abelian groups.

**Theorem 5.** *Any semidirect product group  $\mathcal{G} \simeq \mathcal{K} \otimes_s \mathcal{N}$  is a subgroup of a group homomorphic to the group of automorphisms of  $\mathcal{N}$  [13].*

### 6.1. Mackey theorems for the representations of semidirect product groups

The Mackey theorems are valid for a general class of topological groups but we will only require the more restricted case  $\mathcal{G} \simeq \mathcal{K} \otimes_s \mathcal{N}$  where the group  $\mathcal{G}$  and subgroups  $\mathcal{K}, \mathcal{N}$  are smooth Lie groups. The central extension of any connected Lie group is simply connected and therefore generally has the form of a semidirect product due to Theorem 3 (Levi). Theorem 5 further constrains the possible homogeneous groups  $\mathcal{K}$  of the semidirect product given the normal subgroup  $\mathcal{N}$ .

The first Mackey theorem is the induced representation theorem that gives a method of constructing a unitary representation of a group (that is not necessarily a semidirect product group)

from a unitary representation of a closed subgroup. The second theorem gives a construction of certain representations of a certain subgroup of a semidirect product group from which the complete set of unitary irreducible representations of the group can be induced. This theorem is valid for the general case where the normal subgroup  $\mathcal{N}$  is a nonabelian group. In the special case where the normal subgroup  $\mathcal{N}$  is abelian, the theorem may be stated in a simpler form.

**Theorem 6. (Mackey).** Induced representation theorem. *Suppose that  $\mathcal{G}$  is a Lie group and  $\mathcal{H}$  is a Lie subgroup,  $\mathcal{H} \subset \mathcal{G}$  such that  $\mathbb{K} \simeq \mathcal{G}/\mathcal{H}$  is a homogeneous space with a natural projection  $\pi : \mathcal{G} \rightarrow \mathbb{K}$ , an invariant measure and a canonical section  $\Theta : \mathbb{K} \rightarrow \mathcal{G} : k \mapsto g$  such that  $\pi \circ \Theta = \text{Id}_{\mathbb{K}}$  where  $\text{Id}_{\mathbb{K}}$  is the identity map on  $\mathbb{K}$ . Let  $\rho$  be a unitary representation of  $\mathcal{H}$  on the Hilbert space  $\mathbf{H}^\rho$ :*

$$\rho(h) : \mathbf{H}^\rho \rightarrow \mathbf{H}^\rho : |\varphi\rangle \mapsto |\tilde{\varphi}\rangle = \rho(h) |\varphi\rangle, \quad h \in \mathcal{H}.$$

Then a unitary representation  $\varrho$  of a Lie group  $\mathcal{G}$  on the Hilbert space  $\mathbf{H}^\varrho$ ,

$$\varrho(g) : \mathbf{H}^\varrho \rightarrow \mathbf{H}^\varrho : |\psi\rangle \mapsto |\tilde{\psi}\rangle = \varrho(g) |\psi\rangle, \quad g \in \mathcal{G},$$

may be induced from the representation  $\rho$  of  $\mathcal{H}$  by defining

$$\tilde{\psi}(k) = (\varrho(g)\psi)(k) = \rho(g^\circ)\psi(g^{-1}k), \quad g^\circ = \Theta(k)^{-1}g\Theta(g^{-1}k), \quad (148)$$

where the Hilbert space on which the induced representation  $\varrho$  acts is given by  $\mathbf{H}^\varrho \simeq \mathbf{L}^2(\mathbb{K}, \mathbf{H}^\rho)$  [14], [13].

The proof is straightforward given that the section  $\Theta$  exists by showing first that  $g^\circ \in \ker(\pi) \simeq \mathcal{H}$  and therefore  $\rho(g^\circ)$  is well defined.

**Definition 5. (Little groups):** Let  $\mathcal{G} = \mathcal{K} \otimes_s \mathcal{N}$  be a semidirect product. Let  $[\xi] \in U_{\mathcal{N}}$  where  $U_{\mathcal{N}}$  denotes the unitary dual whose elements are equivalence classes of unitary representations of  $\mathcal{N}$  on a Hilbert space  $\mathbf{H}^\xi$ . Let  $\rho$  be a unitary representation of a subgroup  $\mathcal{G}^\circ = \mathcal{K}^\circ \otimes_s \mathcal{N}$  on the Hilbert space  $\mathbf{H}^\xi$  such that  $\rho|_{\mathcal{N}} = \xi$ . The little groups are the set of maximal subgroups  $\mathcal{K}^\circ$  such that  $\rho$  exists on the corresponding stabilizer  $\mathcal{G}^\circ \simeq \mathcal{K}^\circ \otimes_s \mathcal{N}$  and satisfies the fixed point equation

$$\hat{\varsigma}_{\rho(k)}[\xi] = [\xi], \quad k \in \mathcal{K}^\circ. \quad (149)$$

In this definition the dual automorphism is defined by

$$(\hat{\varsigma}_{\rho(g)}\xi)(h) = \rho(g)\rho(h)\rho(g)^{-1} = \rho(ghg^{-1}) = \xi(\varsigma_g h) \quad (150)$$

for all  $g \in \mathcal{G}^\circ$  and  $h \in \mathcal{N}$ . The equivalence classes of the unitary representations of  $\mathcal{N}$  are defined by

$$[\xi] = \{\hat{\varsigma}_{\xi(h)}\xi | h \in \mathcal{N}\}. \quad (151)$$

A group  $\mathcal{G}$  may have multiple little groups  $\mathcal{K}^\circ_\alpha$  whose intersection is the identity element only. We will generally leave the label  $\alpha$  implicit.

**Theorem 7. (Mackey).** Unitary irreducible representations of semidirect products. *Suppose that we have a semidirect product Lie group  $\mathcal{G} \simeq \mathcal{K} \otimes_s \mathcal{N}$ , where  $\mathcal{K}, \mathcal{N}$  are Lie subgroups. Let  $\xi$  be the unitary irreducible representation of  $\mathcal{N}$  on the Hilbert space  $\mathbf{H}^\xi$ . Let  $\mathcal{G}^\circ \simeq \mathcal{K}^\circ \otimes_s \mathcal{N}$  be a maximal stabilizer on which there exists a representation  $\rho$  on  $\mathbf{H}^\xi$  such that  $\rho|_{\mathcal{N}} = \xi$ . Let  $\sigma$  be a unitary irreducible representation of  $\mathcal{K}^\circ$  on the Hilbert space  $\mathbf{H}^\sigma$ . Define the representation*

$\varrho^\circ = \sigma \otimes \rho$  that acts on the Hilbert space  $\mathbf{H}^{\varrho^\circ} \simeq \mathbf{H}^\sigma \otimes \mathbf{H}^\xi$ . Determine the complete set of stabilizers and representations  $\rho$  and little groups that satisfy these properties, that we label by  $\alpha, \{(\mathcal{G}^\circ, \varrho^\circ, \mathbf{H}^{\varrho^\circ})_\alpha\}$ . If for some member of this set  $\mathcal{G}^\circ \simeq \mathcal{G}$  then for this case the representations are  $(\mathcal{G}, \varrho, \mathbf{H}^\varrho) \simeq (\mathcal{G}^\circ, \varrho^\circ, \mathbf{H}^{\varrho^\circ})$ . For the cases where the stabilizer  $\mathcal{G}^\circ$  is a proper subgroup of  $\mathcal{G}$  then the unitary irreducible representations  $(\mathcal{G}, \varrho, \mathbf{H}^\varrho)$  are the representations induced (using Theorem 6) by the representations  $(\mathcal{G}^\circ, \varrho^\circ, \mathbf{H}^{\varrho^\circ})$  of the stabilizer subgroup. The complete set of unitary irreducible representations is the union of the representations  $\cup_\alpha \{(\mathcal{G}, \varrho, \mathbf{H}^\varrho)_\alpha\}$  over the set of all the stabilizers and corresponding little groups.

This major result and its proof are due to Mackey[14]. Our focus in this paper is on applying this theorem.

#### 6.1.1. Abelian normal subgroup

The theorem simplifies for special cases where the normal subgroup  $\mathcal{N}$  is an abelian group,  $\mathcal{N} \simeq \mathcal{A}(n)$ . An abelian group has the property that its unitary irreducible representations  $\xi$  are the characters acting on the Hilbert space  $\mathbf{H}^\xi \simeq \mathbb{C}$ ,

$$\xi(a) |\phi\rangle = e^{ia \cdot v} |\phi\rangle, \quad a, v \in \mathbb{R}^n. \quad (152)$$

The unitary irreducible representations are labeled by the  $v_i$  that are the eigenvalues of the hermitian representation of the basis  $\{A_i\}$  of the abelian Lie algebra,

$$\hat{A}_i |\phi\rangle = \xi'(A_i) |\phi\rangle = v_i |\phi\rangle. \quad (153)$$

The equivalence classes  $[\xi] \in \mathcal{U}_{\mathcal{A}(n)}$  each have a single element  $[\xi] \simeq \xi$  as, for the abelian group, the expression (151) is trivial. The representations  $\rho$  act on  $\mathbf{H}^\xi \simeq \mathbb{C}$  and are one dimensional and therefore must commute with the  $\xi$ . Therefore, in equation (150),  $\rho(g)\xi(h)\rho(g)^{-1} = \xi(h)$  and (149) simplifies to

$$\xi(a) = \xi(\varsigma_k a) = \xi(kak^{-1}), a \in \mathcal{A}(m), \quad k \in \mathcal{K}^\circ. \quad (154)$$

**Theorem 8. (Mackey).** Unitary irreducible representations of a semidirect product with an abelian normal subgroup. Suppose that we have a semidirect product group  $\mathcal{G} \simeq \mathcal{K} \otimes_s \mathcal{A}$  where  $\mathcal{A}$  is abelian. Let  $\xi$  be the unitary irreducible representation (that are the characters) of  $\mathcal{A}$  on  $\mathbf{H}^\xi \simeq \mathbb{C}$ . Let  $\mathcal{K}^\circ \subseteq \mathcal{K}$  be a Little group defined by (154) with the corresponding stabilizers  $\mathcal{G}^\circ \simeq \mathcal{K}^\circ \otimes_s \mathcal{A}$ . Let  $\sigma$  be the unitary irreducible representations of  $\mathcal{K}^\circ$  on the Hilbert space  $\mathbf{H}^\sigma$ . Define the representation  $\varrho^\circ = \sigma \otimes \xi$  of the stabilizer that acts on the Hilbert space  $\mathbf{H}^{\varrho^\circ} \simeq \mathbf{H}^\sigma \otimes \mathbb{C}$ . The theorem then proceeds as in the case of the general Theorem 7.

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